Final is on

§B Fractals

Characterized by self-similarity to an infinite scale

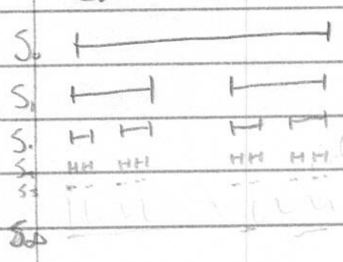
→ Mandelbrot's "Fractal Geometry of Nature"

Def fractals:
Complex objects with finite structure of arbitrary small scales, usually exhibits self-similarity

β
$\beta\beta\beta$
$\beta\alpha\alpha$
$\alpha\alpha\alpha$
$\alpha\alpha\alpha$

Countability
$|E| = |\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| << |\mathbb{R}|$

Cantor Set



$\boxed{d = 0.63}$ $= \dfrac{\ln 2}{\ln 3}$

Dimension d is the minimum number of parameters to describe any point on the object.

$\mathbb{N}$
$\mathbb{N}$
$\mathbb{N}$

$m=1$, $r=1$ $\quad$ $m=4$, $r=2$ $\quad$ $m=9=3^{2}$, $r=3$ — $d=2$

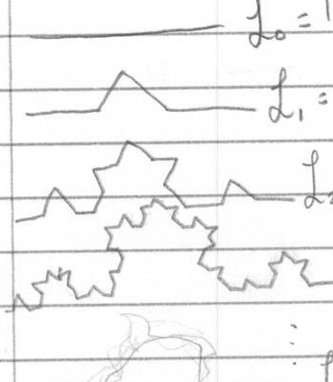
Scaling factor to fit m copies

$\boxed{m = r^d \text{ defines dimension}}$

$$d = \frac{\log m}{\log r}$$

Van Koch

$r=3$
$m=4$



$\mathcal{L}_0 = 1$

$\mathcal{L}_1 = \frac{4}{3}\mathcal{L}_0$

$\mathcal{L}_2 = \frac{4}{3}\mathcal{L}_1 = \left(\frac{4}{3}\right)^2 \mathcal{L}_0$

$\vdots$

$\mathcal{L}_n = \left(\frac{4}{3}\right)^n \mathcal{L}_0$

$m = 4^n$
$r = 3^n$

$d = \dfrac{\ln 4}{\ln 3} = \boxed{1.26}$

Infinite space, but can fit in a finite interval

Math 3341 (Spring 2015): Final Exam

Tuesday April 28th, 8:00 AM - 10:00 AM

Dr. Ramis Movassagh

Your Name :

Problem	1	2	3	4	5	6	TOTAL
Grade							

- You might find these formulas helpful:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\cos^2 x = \frac{1+\cos 2x}{2} \qquad \sin^2 x = \frac{1-\cos 2x}{2} \qquad 2\sin x \cos x = \sin 2x$$

- You might find these formulas for two time-scale perturbation theory useful:

$$\tau \equiv t$$
$$T \equiv \epsilon t$$

and the derivatives with respect to t get transformed to

$$\dot{x} = \partial_\tau x_0 + \epsilon\left(\partial_T x_0 + \partial_\tau x_1\right) + \mathcal{O}\left(\epsilon^2\right)$$
$$\ddot{x} = \partial_{\tau\tau} x_0 + \epsilon\left(\partial_{\tau\tau} x_1 + 2\partial_{T\tau} x_0\right) + \mathcal{O}\left(\epsilon^2\right)$$

1. (20 pts.) a) Find all the fixed points of $\dot{x} = ax - x^3$, $a > 0$

f.p.: $ax - x^3 = 0 \quad x(a - x^2) = x(\sqrt{a} + x)(\sqrt{a} - x) = 0$

$x^* = 0$

$x^* = \pm\sqrt{a}$

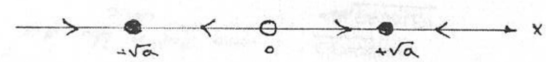
b) Use Linear Stability analysis to determine the stability of each of the fixed points in part a.

$f(x) \equiv ax - x^3 \qquad f'(x) = a - 3x^2$

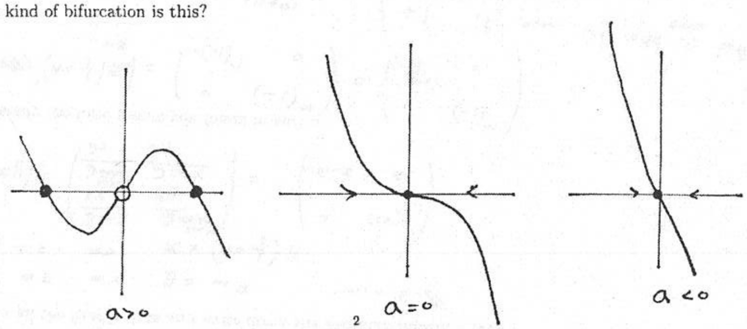
$f'(0) = a > 0$ UNSTABLE

$f'(\pm\sqrt{a}) = a - 3a = -2a < 0$ stable.

c) Sketch the one-dimensional flow based on what you found in parts a and b.



d) Now suppose we vary a from $a < 0$ to $a = 0$ to $a > 0$. What is (are) the critical point(s). If any, what kind of bifurcation is this?



$a = 0$ Bifurcation Pt.

PITCHFORK BIFURCATION.

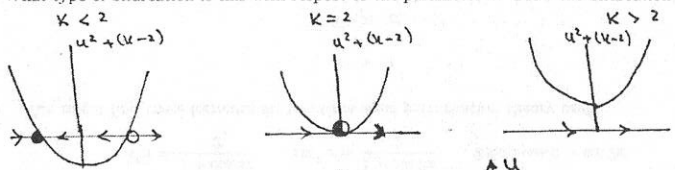
2. (20 pts.) Consider $\dot{u} = k - 2\cos u$ where $k > 0$ and u is small in magnitude.

a) In this limit find the fixed points

$|u| \ll 1 \Rightarrow \cos u = 1 - u^2/2$

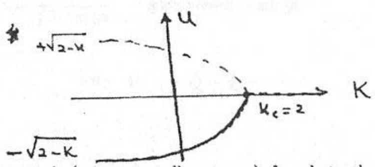
$\dot{u} = (K-2) + u^2 \qquad u^* : \quad u = \pm\sqrt{2-K} \qquad K < 2$ better

b) What type of bifurcation is this with respect to the parameter k? Draw the bifurcation diagram.



Saddle node bifurcation

Bifurcation Diagram:



c) Suppose we consider the same flow but on a circle (u is not small anymore), for what values of k are there fixed points on the circle?

f.p. $\quad \dot{u} = 0 \Rightarrow K - 2\cos u = 0 \Rightarrow \cos u = K/2$

$|K| \leq 2 \qquad$ i.e. $\quad K \in [-2, 2]$

3. (20 pts.) Consider the dynamical system $\dot{x} = \sin y$ and $\dot{y} = \cos x$.

a) Find all the fixed points and write down the Jacobian matrix $J(x, y)$

$m, n \in \mathbb{Z}$

$\dot{x} = 0 \Rightarrow y = m\pi$

$\dot{y} = 0 \Rightarrow x = (n + \frac{1}{2})\pi$

$$J(x,y) = \begin{pmatrix} \frac{\partial \sin y}{\partial x} & \frac{\partial \sin y}{\partial y} \\ \frac{\partial \cos x}{\partial x} & \frac{\partial \cos x}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & \cos y \\ -\sin x & 0 \end{pmatrix}$$

b) Classify the fixed points you found in part a

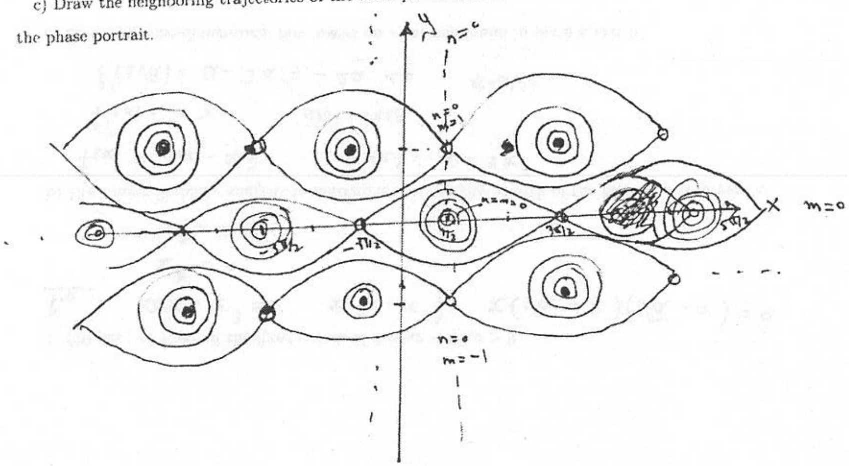
$$J\left((n+\tfrac{1}{2})\pi, m\pi\right) = \begin{pmatrix} 0 & (-1)^m \\ -(-1)^n & 0 \end{pmatrix} = \begin{pmatrix} 0 & (-1)^m \\ (-1)^{n+1} & 0 \end{pmatrix}$$

$\tau = 0 \qquad \Delta = (-1)^{n+m} = \begin{cases} +1 & \text{if } n, m \text{ both odd or both even} \\ -1 & \text{o/w} \end{cases}$

$\therefore$ n, m both odd or both even : CENTERS

o/w : SADDLES

c) Draw the neighboring trajectories of the fixed points and fill in the rest to get a representation of the phase portrait.



4. (20 pts.) Consider the strange motion given by $\ddot{u} = \dot{u} + 2\sin u$

a) Find all the fixed points

$$\dot{u} = v$$
$$\dot{v} = v + 2\sin u$$

$v^* = 0$

$u^* = m\pi \quad m \in \mathbb{Z}$

b) Write down the equation of the motion near the origin, subjected to initial conditions $u(0) = 0$ and $\dot{u}(0) = 1$.

Linear Stability Analysis gives

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = J(u,v) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2\cos u & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Near $(0,0)$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$\lambda_1 = 2 \qquad \vec{v}_{\lambda=2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\lambda_2 = -1 \qquad \vec{v}_{\lambda=-1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\begin{pmatrix} u \\ v \end{pmatrix}(t) = C_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \overset{\text{i.c.}}{=} \begin{pmatrix} u \\ v \end{pmatrix}(0) = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$C_1 + C_2 = 0$

$-C_1 + 2C_2 = 1$

$C_1 = -1/3 \qquad C_2 = 1/3$

$$\begin{pmatrix} u \\ v \end{pmatrix}(t) = -\frac{1}{3} e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{3} e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

5. (20 pts.) Find the lowest-order approximate solution to

$$\ddot{x} + 3x + \epsilon x^2 = 0$$

$$x(0) = 0 \quad , \quad \dot{x}(0) = 1 \quad .$$

where $\epsilon \ll 1$.

$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$

$$\partial_{\tau\tau} x_0 + \epsilon(\partial_{\tau\tau} x_1 + 2\partial_{T\tau} x_0) + 3(x_0 + \epsilon x_1 + \cdots)$$
$$+ \epsilon \underbrace{(x_0 + \epsilon x_1 + \cdots)^2}_{= x_0^2 + \mathcal{O}(\epsilon)} = 0$$

$O(1)$: $\partial_{\tau\tau} x_0 + 3x_0 = 0 \qquad x_0 = A(T)\sin\sqrt{3}\tau + B(T)\cos\sqrt{3}\tau$

$O(\epsilon)$: $\partial_{\tau\tau} x_1 + 3x_1 = -2\left[A'\sqrt{3}\cos(\sqrt{3}\tau) - B'\sqrt{3}\sin\sqrt{3}\tau\right]$
$$- \underbrace{\left[A^2\sin^2\sqrt{3}\tau + B^2\cos^2\sqrt{3}\tau + AB\sin(2\sqrt{3}\tau)\right]}_{\text{Net secular}}$$

RHS : $-2\partial_{T\tau} x_0 - x_0^2$

Secular terms $-2\sqrt{3}A'\cos\sqrt{3}\tau + 2\sqrt{3}B'\sin\sqrt{3}\tau = 0$

force to zero

$A' = B' = 0$: $A = A_0 \qquad B = B_0$

$x_0(\tau, T) = A_0 \sin\sqrt{3}\tau + B_0\cos\sqrt{3}\tau$

$x_0(0) = x(0) = 0 \implies B_0 = 0$

$x_0'(0) = \dot{x}(0) = \sqrt{3}A_0 \cos\sqrt{3}\tau\big|_0 = \sqrt{3}A_0 = 1 \quad \therefore A_0 = \frac{1}{\sqrt{3}}$

$$\boxed{x_0(t) = \frac{1}{\sqrt{3}} \sin(\sqrt{3}\,t)}$$

6. (20 pts.) Show that the system $\dot{x} = x - y - x^3$, $\dot{y} = x + y - y^3$ has a periodic solution. [Hint: (0, 0) is the only fixed point (you don't need to show this). Start with $r^2 = x^2 + y^2$ and find a trapping region by identifying when $\dot{r} > 0$ and $\dot{r} < 0$]

$r^2 = x^2 + y^2 \qquad \cancel{2} r\dot{r} = \cancel{2} x\dot{x} + \cancel{2} y\dot{y}$

$$= x(x - \cancel{y} - x^3) + y(\cancel{x} + y - y^3)$$

$$= x^2 - x^4 + y^2 - y^4 = \underbrace{x^2 + y^2}_{r} - x^4 - y^4$$

$$\dot{r} = r - \frac{1}{r}(x^4 + y^4) = r - r^3(\cos^4\theta + \sin^4\theta)$$

$$\cos^4\theta + \sin^4\theta = \cos^2\theta(1 - \sin^2\theta) + \sin^2\theta(1 - \cos^2\theta)$$

$$= 1 - 2\sin^2\theta\cos^2\theta = 1 - \frac{1}{2}\sin^2(2\theta)$$

$$\therefore \frac{1}{2} \le \cos^4\theta - \sin^4\theta \le 1$$

$$\boxed{r - r^3 \le \dot{r} \le r - \frac{1}{2}r^3}$$

$r = \epsilon \ll 1 \Rightarrow \epsilon \le \dot{r} \Rightarrow \dot{r} > 0$ spirals out from (0,0)

$\dot{r} < 0$ if $r - \frac{1}{2}r^3 < 0 \Rightarrow r\left(1 - \frac{r^2}{2}\right) < 0 \Rightarrow 1 - \frac{r^2}{2} < 0$

$\Rightarrow \boxed{r > \sqrt{2}}$

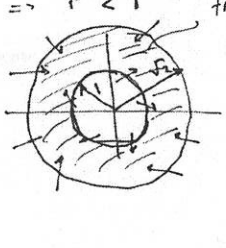
$\boxed{\dot{r} < 0 \text{ if } r > \sqrt{2}}$

We can get a better ring by $\dot{r} > 0$ for r:

$\dot{r} \ge r - r^3 \Rightarrow \dot{r} > 0$ if $r - r^3 > 0$

$r(1 - r^2) > 0 \Rightarrow 1 - r^2 > 0 \Rightarrow r^2 < 1$

trapping Region



Poincare - Bendixson theorem

Guarantees a periodic solut inside.